for the solution of problem (1)—(4). Namely

$$\varepsilon_{ij}(u) = \frac{\partial}{\partial\sigma_{ij}}\left(\frac{1}{2}c_{klpq}\sigma_{kl}\sigma_{pq} + \frac{2}{n+1}B(t)I(s)^{n+1}\right)$$

Note that the Signorini boundary value problems considered in this paper describe the case when the possible area of contact is selected in advance and cannot grow with time (i.e., the greatest possible area of contact is selected), although the presence of contact at any given point is not assumed in advance but is determined only as a result of solving the problem.

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NON-LINEAR DEFORMATIONS AND LIMIT EQUILIBRIUM OF THREE-DIMENSIONAL CURVILINEAR RODS*

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The stress and deformation state of three-dimensional curvilinear rods of circular cross-section is investigated byond the elastic limit. The Kirchhoff-Love hypotheses used. The rod defromations are assumed to be small, but the displacements and the angles of rotation of the central line are arbitrary. The relation between the deformation and the stress states in the plastic region of the material is taken in the form of a linear relation /1/ between the deformation rates and stresses. The coefficients of the equations of this connection are assumed to be specified (for example in the form of a table) by functions of stress components, deformations, time, temperatures, etc. An appropriate selection of these coefficients enables one to describe various models of a solid deformable body.

The method of linearizing the resolving system of equations proposed here enables us to use, for solving specific problems, computational algorithms developed in investigations of geometrically non-linear deformations of elastic rods. It is shown that under specific conditions the elastic kernel, whose cross-section is of elliptic form, degenerates either into a point or a line, and the rod cross-section passes into a purely plastic state. In the purely plastic state the relation between the moments and the force acting over the cross-section is finite, which in the space of generalized force factors (the dimensionless axial force, the twisting and bending moments) are fairly accurately approximated by a sphere. The application

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of the Mises principle leads to relations of the type of an associated law. Numerical solutions are presented for a number of problems for an elastic-plastic and, also, for a rigid-plastic material.

1. **Basic equations of elastic-plastic deformations.** Consider a three-dimensional curvilinear rod of circular cross-section of radius R whose middle line is defined by the equation $\mathbf{r} = \mathbf{r}(s)$, where s is the length of the arc of the middle line.

We define in the rod cross-section normal to its middle line an orthogonal system of Cartesian coordinates x, y, z with unit vectors $\mathbf{i}, \mathbf{j}\ \boldsymbol{\tau}$. The unit vector $\boldsymbol{\tau}$ is directed along the tangent to the middle line so that $d\mathbf{r}/ds = \boldsymbol{\tau}$, and the direction of the $\mathbf{i}, \mathbf{j}$ axes that lie in the cross-section plane can be selected arbitrarily (they may coincide or make a specific angle with the intrinsic trihedron axes). The radius vector of an arbitrary point of the rod can now be represented in the form

$$\mathbf{e}(x, y, s) = \mathbf{r}(s) + x\mathbf{i} + y\mathbf{j}$$

We introduce for the orthogonal basis $\mathbf{i}, \mathbf{j}, \boldsymbol{\tau}$ the Darboux vector $\mathbf{k}(s)$ which is defined by the equations

$$d\mathbf{i}/ds = [\mathbf{k}(s) \times \mathbf{i}(s)],\ d\mathbf{j}/ds = [\mathbf{k}(s) \times \mathbf{j}(s)],\ d\boldsymbol{\tau}/ds = [\mathbf{k}(s) \times \boldsymbol{\tau}(s)]$$

and shows by what amount the trihedron rotates on transferring from one point of the rod axis to another infinitely close point.

Let us consider the deformed state, denoting all respective quantities by a prime. By the Kirchhoff-Love hypotheses, the unit vectors $\mathbf{i}', \mathbf{j}', \boldsymbol{\tau}'$ form, as before, an orthogonal basis, and any arbitrary points of the rod cross-section that, prior to deformation, have in the basis $\mathbf{i}, \mathbf{j}, \boldsymbol{\tau}$ coordinates x, y, retain them in the basis $\mathbf{i}', \mathbf{j}', \boldsymbol{\tau}'$.

Denoting by $\mathbf{k}'$ the Darboux vector of the triheron $\mathbf{i}', \mathbf{j}', \boldsymbol{\tau}'$, for the deformed state we have

$$d\mathbf{r}/ds = \boldsymbol{\tau}' \qquad (1.1)$$

$$d\mathbf{i}'/ds = [\mathbf{k}' \times \mathbf{i}'],\quad d\mathbf{j}'/ds = [\mathbf{k}' \times \mathbf{j}'],\quad d\boldsymbol{\tau}/ds = [\mathbf{k}' \times \boldsymbol{\tau}'] \qquad (1.2)$$

We introduce the relative deformation vector of the middle line of the rod

$$\boldsymbol{\varkappa} = \mathbf{k}' - \mathbf{k} \qquad (1.3)$$

with components $\varkappa_x = \varkappa_1$, $\varkappa_y = \varkappa_2$, $\varkappa_z = \varkappa_3$, that defines the difference of the angle of rotation of two infinitely close cross sections resulting from the deformation of the middle line of the rod.

Using the Kirchhoff-Love hypotheses, we can express the deformation components of arbitrary points of the rod cross-section in terms of the components of the vector $\boldsymbol{\varkappa}$ as

$$\varepsilon_z = \varepsilon_1 = \varepsilon_0 + y\varkappa_x + x\varkappa_y;\quad \gamma_{xz} = \varepsilon_2 = -\varkappa_z y/2;\quad \gamma_{yz} = \varepsilon_3 = \varkappa_z x/2 \qquad (1.4)$$

Henceforth we shall call the components of the vector $\boldsymbol{\varkappa}$ and the magnitude of the relative length of the middle line of the rod $\varepsilon_0 = \varkappa_4$ the curvature (deformation) parameters of the middle line.

Substituting deformations (1.4) into Hooke's law, for the stress components we obtain

$$\sigma_z = \sigma_1 = \mathrm{E}(\varepsilon_0 + y\varkappa_x + x\varkappa_y),\quad \tau_{xz} = \sigma_2 = -G\varkappa_z y,\quad \tau_{yz} = \sigma_3 = G\varkappa_z x \qquad (1.5)$$

where E and G are the moduli of elasticity of the first and second kind.

The stress state of the rod may be defined by the resulting quantities related to the whole rod thickness: the moments and forces expressed in terms of the stress components in the form

$$M_x = M_1 = \iint \sigma_z y\, dF,\quad M_y = M_2 = \iint \sigma_z x\, dF \qquad (1.6)$$

$$N_z = M_4 = \iint \sigma_z\, dF,\quad M_z = M_3 = \iint (\tau_{yz} x - \tau_{xz} y)\, dF$$

These quantities must satisfy the equilibrium equations which can be conveniently written in the vector form /2/

$$d\mathbf{M}/ds = [\mathbf{N} \times \boldsymbol{\tau}'] + \mathbf{m},\quad d\mathbf{N}/ds' = \mathbf{q} \qquad (1.7)$$

where $\mathbf{m}$ and $\mathbf{q}$ are, respectively, the vectors of the external distributed moment and external load.

Substituting the stress components (1.5) into (1.6), we obtain

$$M_i = J_{ij}^{e}\varkappa_j,\quad i, j = 1, \ldots, 4 \qquad (1.8)$$

The matrix of elastic stiffnesses is diagonal and constant, and its coefficients have the form $J_{11}^e = J_{22}^e = JE$, $J_{33}^e = GJ_p$, $J_{44} = EF$, where $J = \pi R^4/4$, $J_p = \pi R^4/2$ are the moments of intertia, and $F = \pi R^2$ is the area of cross-section of the rod.

Equations (1.1), (1.3), (1.8) completely define the stress-strain state of an elastic rod, and can be used to obtain the usual Clebsh-Kirchoff-Love equations /3/.

Substituting the stress components (1.5) into the Mises plasticity conditions, we obtain the equation

$$^1/_3E^2(\varepsilon_0 + \varkappa_x y + \varkappa_y x)^2 + G^2\varkappa_z^2(x^2 + y^2) = \tau_s^2 \tag{1.9}$$

that defines in the xy plane an ellipse whose center does not coincide with the centre of gravity of the cross-section, when $\varepsilon_0 \neq 0$. As long as the deformations are small, the rod cross-section is in the elastic state and, consequently, lies entirely inside the ellipse (1.9). Assuming that all the distortion parameters are positive, plasticity first appears on the contour of the rod at a point with coordinates

$$x = \frac{R\varkappa_y}{\sqrt{\varkappa_x^2 + \varkappa_y^2}}, \quad y = \frac{R\varkappa_x}{\sqrt{\varkappa_x^2 + \varkappa_y^2}}$$

and from Eq.(1.9) it follows that the distorition parameters satisfy the condition

$$(\xi + \sqrt{\eta_x^2 + \eta_y^2})^2 + \zeta^2 = 1 \tag{1.10}$$

where we have introduced the distortion parameters

$$\xi = \frac{\varepsilon_0 E}{\sqrt{3}\,\tau_s}, \quad \eta_x = \frac{R\varkappa_x E}{\sqrt{3}\,\tau_s}, \quad \eta_y = \frac{R\varkappa_y E}{\sqrt{3}\,\tau_s}, \quad \zeta = \frac{R\varkappa_z G}{\tau_s}$$

In space $\xi, \eta_x, \eta_y, \zeta$ Eq.(1.10) defines a surface whose intersection by the deformation trajectory of the rod cross-section passes from the elastic to the elastic-plastic state.

For a fairly wide class of models of the mechanics of a solid deformable body (an ideally plastic body, and elastic-plastic reinforcing material, a visco-plastic-elastic material, etc.) the relation between the deformation and stress tensor components, outside the elastic limits, with the notation introduced in (1.4), (1.5), may be represented in the form

$$\sigma_i^{\cdot} = a_{ij}\varepsilon_j^{\cdot} + b_i, \quad i, j = 1, 2, 3 \tag{1.11}$$

Here and subsequently the recurring subscripts indicate summation, and a dot denotes differentiation with respect to time or some other monotonically changing parameter defining the development of the process.

The equations linking the stressed and deformed state of the material of the form of (1.11) was first used in /1/, when investigating the non-linear deformations of shells of revolution. A method of deriving the coefficients a_{ij}, b_j for the deformation theory of plasticity, the theory of plastic flow with isotropic and translational reinforcement, of the generalized Maxwell model, etc. was also described.

Assuming the deformation process to be monotonic and differentiating relations (1.5), (1.6) with respect to time, we obtain the equations

$$M_i^{\cdot} = J_{ij}^{\cdot}\varkappa_j^{\cdot} + B_i, \quad i, j = 1, \ldots, 4 \tag{1.12}$$

that link the time derivatives of the axial force and moment components with the time derivatives of the deformation parameters.

The form of the coefficients of the matrix $\|J\|$ is shown by the first row of that matrix

$$J_{11} = \iint\limits_{F^e} Ey^2\,dF + \iint\limits_{F^p} a_{11}y^2\,dF, \quad J_{12} = E\iint\limits_{F^e} xy\,dF + \iint\limits_{F^p} a_{11}xy\,dF \tag{1.13}$$

$$J_{13} = \iint\limits_{F^p} (a_{12}y^2 + a_{13}xy)\,dF, \quad J_{14} = E\iint\limits_{F^e} y\,dF + \iint\limits_{F^p} a_{11}y\,dF$$

$$B_1 = \iint\limits_{F^p} b_1 y\,dF$$

where F^e and F^p are, respectively, the areas of the elastic and plastic region of the rod cross-section.

The system of 20 scalar non-linear differential equations (1.1), (1.2), (1.7), (1.12) containing 20 unknown functions completely define the stress-strain state of a three-dimensional curvilinear rod, and, in general, can only be solved numerically.

The solution of that system may be obtained by "step-by-step linearization" as used when analysing shells /1/ and in /4/ for three-dimensional curvilinear elastic rods. The essence of the method is the splitting of the deformation process into small time steps δt, and calculating for each time layer not the functions required but their increments in time for a rod of the form obtained as a result of the preceding step. To obtain the respective system of resolving differential equations we multiply formulas (1.12) by δt and write it in the form

$$\delta M_i = J_{ij}\delta \varkappa_j + B_i \delta t \tag{1.14}$$

This is follows by differentiation of the remaining 16 equations with respect to t, and again, multiplication by δt.

The system of differential equations thus obtained, which is linear with respect to the increments of the unknown functions, is exactly the same as the system of elastic rods /4/.

With this approach the solution of the problem of elastic deformations of a three-dimensional curvilinear rod differs from the solution of the similar elastic-plastic problem only in the form of the stiffness matrix $\| J \|$. With elastic deformations it is diagonal and its coefficients are constant, while for elastic-plastic deformations (1.14) its coefficients are functions of the deformation parameters and of stress tensor components, and to determine them we have the necessary equations (1.11), (1.13). Note that the problem of evaluating integrals (1.13) can be simplified considerably, if the quantities a_{ij} and b_i in Eqs.(1.11) are independent of the stress components. The feasibility of this representation of the equations of an elastic-plastic material was investigated in detail in /5/. In that case it is necessary to store in the computer memory for the rod cross-sections considered only four values of the deformation parameters that define the deformation components (1.4) at each point of the cross-section. Otherwise it is necessary to store in the computer memory the complete stress component distributions in the plastic regions of the cross-sections considered that are necessary for evaluating integrals (1.13).

2. The limit equilibrium. The theory of the limit equilibrium of three-dimensional curvilinear rods may be obtained independently of the equations of elastic-plastic deformation considered in the preceding section.

To do this let us assume that all rod cross-sections are in the plastic state; the material is ideally rigidly plastic, and obeys the St.Venant-Love-Mises equations of plastic flow. In this case, the stress tensor components are connected with the components of the rates of change of the deformation parameters as follows:

$$\sigma_z = \frac{3\tau_s}{2H}(\varepsilon_0^{\cdot} + \varkappa_x^{\cdot} y + \varkappa_y^{\cdot} x), \quad \tau_{xz} = \frac{\tau_s}{2H}\varkappa_z^{\cdot} y, \quad \tau_{yz} = \frac{\tau_s}{2H}\varkappa_z^{\cdot} x$$
$$2H = [3(\varepsilon_0^{\cdot} + \varkappa_y^{\cdot} x + \varkappa_x^{\cdot} y)^2 + \varkappa_z^{\cdot 2}(x^2 + y^2)]^{1/2} \tag{2.1}$$

where H is the intensity of the shear deformation rate.

Substituting the stress components (2.1) into (1.6), we obtain

$$M_x = 3\tau_s \iint (\varepsilon_0^{\cdot} + \varkappa_x y + \varkappa_y^{\cdot} x)\, y \frac{dF}{H} \tag{2.2}$$
$$M_y = 3\tau_s \iint (\varepsilon_0^{\cdot} + \varkappa_x^{\cdot} y + \varkappa_y^{\cdot} x)\, x \frac{dF}{H}$$
$$M_z = \tau_s \iint \varkappa_z^{\cdot}(x^2 + y^2)\frac{dF}{H}, \quad N_z = 3\tau_s \iint (\varepsilon_0^{\cdot} + \varkappa_x^{\cdot} y + \varkappa_y^{\cdot} x)\frac{dF}{H}$$

We will express the curvature components $\varkappa_x^{\cdot}, \varkappa_y^{\cdot}$ in terms of new quantities

$$\varkappa_x^{\cdot} = \varkappa^{\cdot}\cos\eta, \quad \varkappa_y^{\cdot} = x^{\cdot}\sin\eta \tag{2.3}$$
$$\varkappa^{\cdot} = \sqrt{\varkappa_x^{\cdot 2} + \varkappa_y^{\cdot 2}}, \quad \operatorname{tg}\eta = \varkappa_y^{\cdot}/\varkappa_x^{\cdot} \tag{2.4}$$

and introduce the new system of coordinates x_1, y_1 rotated by an angle η in the positive direction relative to the x, y system of coordinates

$$x_1 = x\cos\eta + y\sin\eta, \quad y_1 = y\cos\eta - x\sin\eta \tag{2.5}$$

Subsituting (2.3) into the first two integrals (2.2) and making the change of variables (2.5), after simple algebra we obtain

$$M_x = M_\nu \sin\eta, \quad M_y = M_\nu \cos\eta \tag{2.6}$$
$$M_\nu = \iint 3\tau_s \varkappa^{\cdot} x_1^2 [3(\varepsilon_0^{\cdot} - \varkappa^{\cdot} x_1)^2 + \varkappa_z^{\cdot 2}(x_1^2 + y_1^2)]^{-1/2} dF \tag{2.7}$$

Note that the curvature components $\varkappa_x^{\cdot}, \varkappa_y^{\cdot}$ appear in expression (2.7) only in terms of their modulus $\varkappa^{\cdot}$ (2.4).

Formulas (2.2), (2.6), (2.7) show that the four quantities M_i depend only on three variables and, consequently, in four-dimensional space form the boundary surface

$$F(M_i) = 0 \tag{2.8}$$

which may be represented in parametric form

$$n=\frac{N_z}{N_z^*}=\frac{\sqrt{3}}{\pi}\iint (x_0-x_1)\frac{dF}{A} \tag{2.9}$$

$$m_\tau=\frac{M_z}{M_z^*}=\frac{3}{2\pi}\iint (x_1^2+y_1^2)\,\mathrm{tg}\,\theta\frac{dF}{A}$$

$$m_\nu=\frac{M_\nu}{M_\nu^*}=\frac{3\sqrt{3}}{4}\iint (x_1-x_0)\,x\frac{dF}{A}$$

$$m_x=m_\nu\sin\eta,\quad m_y=m_\nu\cos\eta$$

$$x_0=\varepsilon_0^{\cdot}/\varkappa^{\cdot},\quad \mathrm{tg}\,\theta=\varkappa_z^{\cdot}/\varkappa_y^{\cdot}$$

$$A=[3(x_1-x_0)^2+\mathrm{tg}^2\theta\,(x_1^2+y_1^2)]^{1/2}$$

$$N^*=\tau_s\sqrt{3}\pi R^2,\quad M_z^*=2\tau_s\pi R^3/3,\quad M_\nu^*=4\tau_s R^3/\sqrt{3}$$

where the asterisk denotes the limit values of the moments and force.

By evaluating the partial derivatives of Eq.(2.8) in the usual manner it can be shown that

$$\varkappa_i^{\cdot}=\lambda\frac{\partial F_i}{\partial M_i} \tag{2.10}$$

where λ is an undetermined multiplier, and there is not summation over the subscript i. It follows from (2.10) that the rate vector of the deformation parameters is orthogonal to the boundary surface of the limit values in the four-dimensional space M_i.

Equations (2.6) show that the intersection of the boundary surface (2.8) by the plane M_x, M_y always forms circles of variable radius M_ν. Hence the complete representation of the boundary surface (2.8) gives its transform in the three-dimensional space n, m_τ, m_ν shown in Fig.1. The lines on this surface correspond to a number of constant values of $y_0=x_0/R$, $\psi_0=2\theta_0/\pi$.

Analysis has shown that the boundary surface may be approximated with sufficient accuracy by the sphere

$$n^2+m_x^2+m_y^2+m_\tau^2=1 \tag{2.11}$$

The sphere intersection with the plane nm_ν is shown in Fig.1 by the dashed line. In the remaining coordinate planes the lines of intersection of surfaces (2.9), (2.11) virtually coincide.

To establish the relation between the deformation parameters and the resulting quantities under conditions (2.11) we use the Mises principle which states /6/ that for given increments of generalized deformations $(\delta\varkappa_z, \delta\varkappa_x, \delta\varkappa_y, \delta\varepsilon_0)$ the generalized stresses (M_z, M_x, M_y, N_z) are

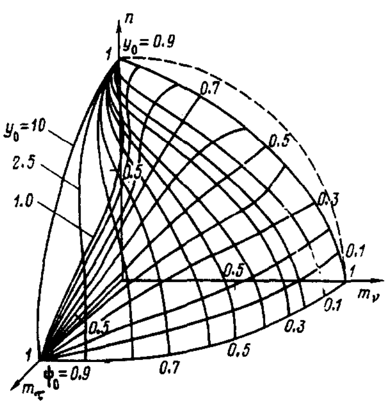

Fig.1

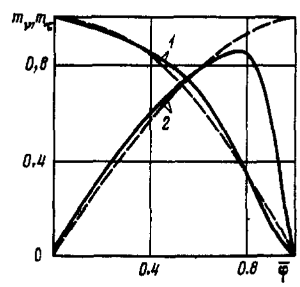

Fig.2

such that the work accomplished by plastic deformation has a stationary value. By determining, in the usual manner, the extremum of the function

$$\delta A=N_z\delta\varepsilon_0+M_x\delta\varkappa_x+M_y\delta\varkappa_y+M_z\delta\varkappa_z$$

subject to condition (2.11), we obtain

$$\delta\varepsilon_0=\delta\lambda\frac{N_z}{N_z^{*2}},\quad \delta\varkappa_x=\delta\lambda\frac{M_x}{M_\nu^{*2}}\quad \delta\varkappa_y=\delta\lambda\frac{M_y}{M_y^{*2}},\quad \delta\varkappa_z=\delta\lambda\frac{M_z}{M_z^{*2}} \tag{2.12}$$

where $\delta\lambda$ is some positive scalar multiplier. These equations show that the use of the Mises principle results in the vector of the increements of the deformation components again being orthogonal to the boundary surface (2.11) defined in the four-dimensional space M_i.

On the basis of (2.11) the quantities M_i are expressed by the three angles ψ, φ, and η in the form

$$M_z = M_z^* \cos\psi \sin\varphi, \quad M_z = M_\nu^* \cos\psi \cos\varphi \sin\eta, \quad M_y = M_\nu^* \cos\psi \cos\varphi \cos\eta \qquad (2.13)$$

Substituting these expressions into (2.12), we obtain

$$\delta\varepsilon_0 = \frac{\delta\lambda}{N_z^*}\sin\psi, \quad \delta\varkappa_y = \frac{\delta\lambda}{M_\nu^*}\cos\psi\cos\varphi\cos\eta \qquad (2.14)$$

$$\delta\varkappa_z = \frac{\delta\lambda}{M_z^*}\cos\psi\sin\varphi, \quad \delta\varkappa_x = \frac{\delta\lambda}{M_\nu^*}\cos\psi\cos\varphi\sin\eta$$

If we assume that in the plastic region the relations of the deformation theory of plasticity hold, then in all calculations in this section the velocities of the deformation components $\varkappa_i^{\cdot}$ or their increments $\delta\varkappa_i^{\cdot}$ must be replaced by the complete deformation components $\varkappa_i$.

3. **Numerical examples**. As an example consider the application of the equations derived above to the problem of changing the helix angle of a circular cylindrical spring.

Let us assume that a circular rod of radius R is wound in the form of a spring with helix angle φ_0 on an absolutely rigid circular cylinder of radius $a - R$ in such a way that its middle line is defined by the equation

$$x = a\cos t, \quad y = a\sin t, \quad z = (a \operatorname{tg}\varphi_0)\, t \qquad (3.1)$$

We have to determine the force factors to be applied to the rod so that its middle line remains as before a helical line (3.1), but with a new helix angle φ_1.

Let us suppose that the deformation is achieved without elongation of the rod axis, and that the line of rod contact with the cylinder remains, as before, the contact line during the whole deformation process.

We will first consider this problem using the equations of limit equilibrium written for the deformation parameters. The system of coordinates x', y', z' is directed along the axes of the natural trihedron so that the x' axis coincides with the principal normal.

The components of the Darboux vector of the deformed state k_x', k_y', k_z' are expressed in terms of the curvature k_1 and twist k_2 of the helical line (3.1)

$$k_x = 0, \quad k_y = k_1\cos^2\varphi_1/a, \quad k_z = k_2 = \sin\varphi_1\cos\varphi_1/a$$

The vector components with respect to the deformation $\varkappa$ are

$$\varkappa_x = 0, \quad \varkappa_y = (\cos^2\varphi_1 - \cos^2\varphi_0)/a, \quad \varkappa_z = (\sin 2\varphi_1 - \sin 2\varphi_0)/(2a) \qquad (3.2)$$

Since the quantities (3.2) are independent of the arc length s, the resulting force factors (2.9) are also constant along the rod axis, and $N_z = M_x = 0$, $\eta = 0$, $M_y = M_\nu$.

Projecting on the x', y' and z' axes the equations of equilibrium (1.7), we find that three of them are satisfied identically, and the remaining three lead to the equations

$$N_y = M_\nu k_z - M_z k_y, \quad N_x = 0, \quad q_x = N_y k_z \qquad (3.3)$$

Substituting (3.2) into (2.2) and noting that

$$x_0 = 0, \quad \operatorname{tg}\theta = -\operatorname{ctg}(\varphi_1 + \varphi_0)$$

we determine the bending moment and torque necessary to obtain the specified deformation trajectory. After this, from (3.3) we determine the shear force N_y and the distributed load of the reaction q_x. Note that in this case integrals (2.9) are evaluated in terms of elliptic integrals of the first and second kind, but in view of their complexity they are not given here.

The solution of the problem is obtained in simple closed form, when the approximate formulas (2.13), (2.14) are used.

Subsituting (3.2) for $\delta\varkappa_y$, $\delta\varkappa_x$ and, also, $\delta\varepsilon_0 = 0$ into (2.14) we obtain

$$\psi = 0, \quad \eta = 0, \quad \operatorname{tg}\varphi = -\frac{\pi\sqrt{3}}{4}\operatorname{ctg}(\varphi_1 + \varphi_0)$$

whence it is possible to determine the angle φ and, then, using (2.13), the bending and twisting moments.

The results obtained the exact formula (2.9) and the approximate formula (2.13) are shown in Fig.2 by the dashed lines for bending a cylindrical spring from an initially rectilinear rod ($\varphi_0 = \pi/2$). They are in good agreement, and agree within the accuracy of the graph in Fig.2, where the reduced moments $m_\nu = M_\nu/M_\nu^*$ (curve *1*) and $m_\tau = M_z/M_z^*$ (curve *2*) are plotted along the ordinate axis and the helix angle φ_1 is shown in fractions of $\pi/2$, i.e. $\bar{\varphi} = 2\varphi_1/\pi$.

The problem can also be solved using the elastic-plastic deformation equations of Sect.2. The only difference is that the deformation parameters (3.2) must be substituted into (1.12) instead of (2.9). The solution of the respective problem for an ideal elastic-plastic material and the deformation theory of plasticity when $\varphi_0 = \pi/2$, $R/a = 10^{-2}$ and $\tau_s/E = 7.2\cdot10^{-3}$ are shown in Fig.2 by the solid lines. They are in good agreement with the results of the theory of limit equilibrium, beginning from the helix angles $\varphi < 0.8\pi/2$. The disagreement observed at $\varphi < 0.8\pi/2$ is explained by the fact that in the region of the $n = 0$ plane the approximating sphere lies inside the surface (2.9). The results are virtually indistinguishable when the accurate relations are used.

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AN APPROXIMATE METHOD OF OPTIMIZING THE SHAPE OF REINFORCEMENT RODS IN NON-UNIFORMLY AGING MATERIALS*

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The problem of optimizing the shape of a rod made of a non-uniformly aging viscoelastic material and reinforced by an elastic material is considered. Geometrical and integral constraints are imposed on the area of cross-section of the rod. The optimum shape is selected to minimize the maximum deflection of the rod in a fixed time interval. An approximate method of optimizing the shape is proposed and justified in the case of slight creep of the material. Results of numerical calculations are presented.

1. Statement of the problem of rod shape optimization. Consider the bending of a rod of length L made from non-uniformly aging viscoelastic material and reinforced by an elastic material. The 0ξ axis is directed along the axis of the rod in the undeformed state. We will denote by $I_0(\xi), I_a, I(\xi)$ the moments of inertia of the cross-sections of the basic material, the reinforcing material, and the whole rod, respectively, and by $S(\xi)$ the rod cross-section at the point ξ. The arrangement of the reinforcement is specified, and is independent of the coordinate ξ. The rod moment of inertia $I(\xi)$ and the area of cross-section $S(\xi)$ are connected by the relation

$$I(\xi) = a_n S^n(\xi) \tag{1.1}$$

where n, a_n are given positive constants. The cross-sectional area of the rod is bounded

$$0 < S_1 \leqslant S(\xi) \leqslant S_2 < \infty \tag{1.2}$$

and the reinforcing material is completely covered by the viscoelastic material. The latter assumption is satisfied for example, when the reinforcement is in the region corresponding to the minimum possible area of cross-section that represents either a rectangle of constant thickness and varying width, or a rectangle of constant width and varying thickness, or a

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